Fluctuation-dissipation theorem and the linear Glauber model

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We obtain exact expressions for the two-time autocorrelation and response functions of the d-dimensional linear Glauber model. Although this linear model does not obey detailed balance in dimensions $d \geqslant 2$, we show that the usual form of the fluctuation-dissipation ratio still holds in the stationary regime. In the transient regime, we show the occurrence of aging, with a special limit of the fluctuation-dissipation ratio, $X_\infty = 1/2$, for a quench at the critical point.

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I. INTRODUCTION

It is widely recognized that the ferromagnetic Ising chain with first-neighbor interactions and Glauber dynamics is one of the simplest, exactly soluble, stochastic dynamical systems [1]. At finite temperatures, in the stationary regime, the two-time spin autocorrelation, $C(t,t')$, and the associated response function, $R(t,t')$, of this Glauber chain are time-translationally invariant, and duly related by the usual expression of the fluctuation-dissipation theorem [2]. There is also an aging regime, with violation of the usual form of the fluctuation-dissipation theorem. At the critical point, at zero temperature, for large values of the observation time t, the fluctuation-dissipation ratio of the Glauber chain assumes the nontrivial limiting value $X_\infty = 1/2$.

We now revisit a linearized version of the Glauber model, proposed by one of us a few years ago [3]. As in the original Glauber model on a d-dimensional hypercubic lattice, we still consider a one-flip stochastic process. Each site $r = 1, \ldots, N$ of the lattice is associated with a spin variable $\sigma_r = \pm 1$. However, the time evolution is now governed by a linear spin-flip ratio,

$$w_r(\sigma) = \frac{\alpha}{2}\left[1 - \frac{\lambda}{2d}\sigma_r \sum_{\delta} \sigma_{r+\delta}\right], \tag{1}$$

where $\lambda \in [0,1]$ is a parameter, the sum is over the $2d$ nearest neighbors of site r, the time scale is set by the parameter α, and σ stands for a configuration of spin variables, $\sigma = \{\sigma_r\}$. The evolution of the probability $P(\sigma,t)$ of the spin configuration σ at time t is given by the master equation

$$\frac{d}{dt}P(\sigma,t) = \sum_{r \in \Lambda} [w_r(\sigma^r)P(\sigma^r,t) - w_r(\sigma)P(\sigma,t)], \tag{2}$$

where σ^r is defined as the configuration σ with σ_r replaced by $-\sigma_r$. From these equations, it is straightforward to calculate exact analytical expression for the site magnetization, $m_r(t) = \langle \sigma_r(t) \rangle$, and the pair correlation function, $q_{r,r'}(t) = \langle \sigma_r(t)\sigma_{r'}(t) \rangle$. In contrast to the original Glauber model, defined by a nonlinear transition rate, with exact solutions restricted to $d=1$, we can now write expressions for $m_r(t)$ and $q_{r,r'}(t)$ in all dimensions d. For $0 \leqslant \lambda < 1$, the linearized model displays a disordered (paramagnetic) phase, with exponentially decaying pair correlations; at the critical point $\lambda = 1$, correlations decay algebraically [3].

It is important to distinguish between this linear Glauber model and the so-called Glauber dynamics, associated with a nonlinear transition rate, and which is used to simulate the Ising model. The linear Glauber model may be regarded as a voter model with noise [4]. It displays just one phase, for all dimensions, as long as noise is finite. In the absence of noise $(\lambda = 1)$ it becomes critical. In one dimension there is a close identification between these two versions of the Glauber model. For $d \geqslant 2$, however, in contrast to the original model, the analytically solvable linear Glauber model is microscopically irreversible (in other words, although having a stationary state, it does not obey detailed balance, and cannot be associated with a Hamiltonian). The fluctuation-dissipation theorem is usually conceived for systems that do obey detailed balance [5,6]. It is then reasonable to ask some questions, including the validity of the fluctuation-dissipation theorem and the presence of an aging regime, about the dynamical behavior of systems that do not obey detailed balance. One of the purposes of this article is to carry out a thorough analytical investigation of a particular system, as the nonlinear Glauber model in $d \geqslant 2$ dimensions, which belongs to the large class of microscopically irreversible models [3,4,7–9].

The dynamical calculations of interest in this investigation are performed in the presence of a (small) perturbation. In the treatment of stochastic models it is natural to introduce the modified one-spin-flip rate,

$$w_r(\sigma) = w_r^0(\sigma) e^{-h_r \sigma_r}, \tag{3}$$

where $w_r^0(\sigma)$ is the unperturbed flipping rate associated with the rth spin, and h_r is the time-dependent disturbance coupled to the dynamic variable σ_r. Alternatively, as the calculations are restricted to small perturbations, we can write

$$w_r(\sigma) = w_r^0(\sigma)(1 - h_r \sigma_r). \tag{4}$$

If the model is microscopically reversible, that is, if the unperturbed transition rate w_r^0 obeys detailed balance, there is a model Hamiltonian $\mathcal{H}_0$, and it is straightforward to show that the perturbed transition rate w_r, given by Eq. (3), also obeys detailed balance. In this reversible case, the model is de-

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scribed by the Hamiltonian $\mathcal{H}=\mathcal{H}_0-\Sigma_r H_r\sigma_r$, where $H_r=h_r/\beta$, and β is proportional to the inverse of the temperature. This form of perturbation is then suitable for reversible models, with a disturbance h_r proportional to the external field. Assuming this expression for the flipping rate, the fluctuation-dissipation theorem is given by

$$R(t,t')=\frac{\partial}{\partial t'}C(t,t'), \tag{5}$$

where

$$R(t,t')=\frac{1}{N}\sum_r \left.\frac{\delta m_r(t)}{\delta h_r(t')}\right|_{h\downarrow 0} \tag{6}$$

is the response function, $m_r(t)$ is the average of σ_r at the observation time t, and

$$C(t,t')=\frac{1}{N}\sum_r \langle \sigma_r(t)\sigma_r(t')\rangle \tag{7}$$

is the autocorrelation function of σ_r between the observation time t and the waiting time t' (with $t\geqslant t'$). For systems obeying detailed balance, we have $h_r=\beta H_r$, and relation (5) reduces to the usual form of the fluctuation-dissipation theorem. Another version of the fluctuation-dissipation theorem relates the susceptibility, associated with a given dynamical variable M, such as the total magnetization, and its variance,

$$\frac{d}{dh}\langle M\rangle=\langle M^2\rangle-\langle M\rangle^2, \tag{8}$$

where h is a static, time-independent disturbance, introduced by the prescription of Eq. (3).

In this work, we show that both forms of the fluctuation-dissipation relation, Eqs. (5) and (8), are valid for the linear Glauber model in the stationary regime. In the transient regime, where aging behavior takes place, these relations are no longer obeyed. It is then appropriate to define [10,11] a fluctuation-dissipation ratio,

$$X(t,t')=\frac{R(t,t')}{\partial C(t,t')/\partial t'}. \tag{9}$$

In the linear Glauber model, for all values of the dimension d, we show that $X(t,t')\rightarrow 1$ in the limit $t'\rightarrow\infty$, except at the critical point, $\lambda=1$, in which case $X(\infty,t')\rightarrow 1/2$.

The layout of this paper is as follows. Some results for the linear Glauber model, including a discussion of the lack of detailed balance for $d\geqslant 2$, and calculations of the site magnetization and the spatial two-body correlations, are reviewed in Sec. II. These one-time functions play a major role in the calculations of the two-time functions, as the autocorrelation and the response functions, which are obtained in Sec. III. In this section, we also consider the stationary limit and make a number of comments on the nonstationary regime. Section IV contains some conclusions.

II. THE LINEAR GLAUBER MODEL

We have already mentioned that the linear Glauber model has been introduced by one of us [3] as an extension of the original Glauber model [1], and may be regarded as a voter model with noise. This linear Glauber model is defined by the (linear) one-spin-flip rate given by Eq. (1), which should be inserted into the master equation (2).

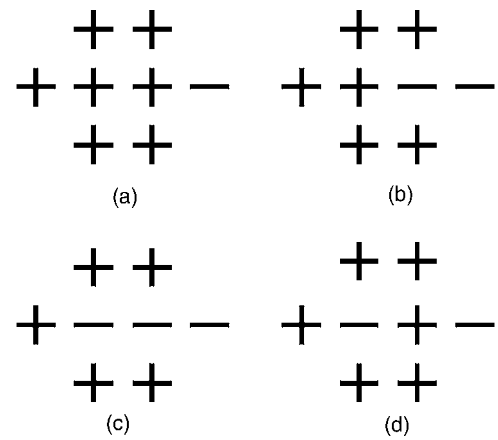

FIG. 1. A possible irreversible sequence.

From Eqs. (1) and (2), it is not difficult to obtain the evolution equations for the site magnetization and the spatial pair correlation,

$$\frac{1}{\alpha}\frac{d}{dt}m_r(t)=-m_r(t)+\frac{\lambda}{2d}\sum_\delta m_{r+\delta}(t) \tag{10}$$

and

$$\frac{1}{\alpha}\frac{d}{dt}q_{r,r'}(t)=-2q_{r,r'}(t)+\frac{\lambda}{2d}\sum_\delta [q_{r,r'+\delta}(t)+q_{r',r+\delta}(t)], \tag{11}$$

for $r\neq r'$. If r and r' are nearest-neighbor sites, the right-hand side of Eq. (11) contains terms like $q_{r,r}(t)$ and $q_{r',r'}(t)$ which should be set equal to 1. The possibility of obtaining these exact expressions, and of performing the exact calculations that are going to be reported in this article, is one of the most relevant features of the linear Glauber model. As shown by Oliveira [3], for $0\leqslant\lambda<1$ this model displays a disordered (paramagnetic) phase with exponentially decaying correlations. For $\lambda\rightarrow 1$, it becomes critical, with algebraically decaying correlations at $\lambda=1$.

The linear Glauber model in one dimension has a reversible dynamics. In one dimension, the probability of occurrence of any sequence of states and the probability of the associated reverse sequence of states are the same. In dimensions larger than one, this is no longer valid. Consider, for instance, the four states shown in Fig. 1, on a square lattice. Suppose that the system follows the sequence of states A, B, C, D, and returns to the initial state A. If the interval Δt between two successive states is small, then, according to the spin-flip rate given by Eq. (3), the probability of occurrence of the sequence $A\rightarrow B\rightarrow C\rightarrow D\rightarrow A$ is given by

$$\begin{aligned} &P(A \to B \to C \to D \to A) \\ &\quad = P(A|D)P(D|C)P(C|B)P(B|A)P(A) \\ &\quad = \frac{1}{16}\left(1 - \frac{\lambda}{2}\right)^2 (1+\lambda)(\alpha\Delta t)^4 P(A). \end{aligned}$$

On the other hand, the reversed sequence, $A \to D \to C \to B \to A$, has the probability

$$\begin{aligned} &P(A \to D \to C \to B \to A) \\ &\quad = P(A|B)P(B|C)P(C|D)P(D|A)P(A) \\ &\quad = \frac{1}{16}\left(1 + \frac{\lambda}{2}\right)^2 (1-\lambda)(\alpha\Delta t)^4 P(A). \end{aligned}$$

These two probabilities are distinct, so that the linear Glauber model on a square lattice is indeed irreversible, the only exception being the trivial case $\lambda=0$. A generalization of this result to larger dimensions is easily found by filling the sites created by the introduction of more dimensions with "+ spins." Hence, the detailed balance cannot be valid, and the stationary state is *a priori* not known. The connection of the transition rates with a Gibbs measure, as has been possible in the one-dimensional case, is now forbidden.

A. Site magnetization

We now introduce the Fourier transform of $m_r(t)$,

$$\tilde{m}_k(t) = \sum_r m_r(t) e^{-irk}, \tag{12}$$

and the Laplace transform of $\tilde{m}_k(t)$,

$$\hat{m}_k(s) = \int_0^\infty dt e^{-st} \tilde{m}_k(t). \tag{13}$$

Using these transforms, the differential equation (10) is reduced to the algebraic form

$$\hat{m}_k(s) = \frac{\hat{m}_k^0}{s + \alpha f(k)}, \tag{14}$$

where

$$f(k) = 1 - \frac{\lambda}{d}\sum_{j=1}^{d} \cos k_j, \tag{15}$$

and $\hat{m}_k^0$ is the Fourier-Laplace transform of $m_r(0)$.

The inverse Laplace transformation leads to

$$\tilde{m}_k(t) = e^{-\alpha f(k) t} \tilde{m}_k(0). \tag{16}$$

Inverse Fourier transforming, we then have

$$m_r(t) = \sum_{r'} \Gamma_{r-r'}(t) m_{r'}(0), \tag{17}$$

where

$$\Gamma_r(t) = \int e^{irk - \alpha f(k) t} \frac{d^d k}{(2\pi)^d}. \tag{18}$$

We can also obtain the site magnetization with reference to an initial time $t=t'$ instead of $t=0$. In this case, we just write

$$m_r(t) = \sum_{r'} \Gamma_{r-r'}(t-t') m_{r'}(t'), \tag{19}$$

where $m_r(t')$ is the site magnetization at time t'.

B. Pair correlation

If we look for translationally invariant solutions of Eq. (11), the spatial correlation between sites r and r' will be a function of distance, $\langle \sigma_r(t)\sigma_{r'}(t)\rangle = q_{r-r'}(t) = q_{r'-r}(t)$. We then write Eq. (11) as

$$\frac{1}{\alpha}\frac{d}{dt} q_r(t) = -2q_r(t) + \frac{\lambda}{d}\sum_\delta q_{r+\delta}(t), \tag{20}$$

for $r \neq 0$, with $q_0(t)=1$ whenever it appears on the right-hand side.

Using a method introduced by Oliveira [3], let us write an equation for $r=0$,

$$\frac{1}{\alpha}\frac{d}{dt} q_0(t) = -2q_0(t) + \frac{\lambda}{d}\sum_\delta q_\delta(t) + b(t), \tag{21}$$

where $b(t)$ is chosen to ensure that $q_0(t)=1$. Actually, $b(t)$ is defined by

$$b(t) = 2 - \frac{\lambda}{d}\sum_\delta q_\delta(t). \tag{22}$$

Consequently, Eqs. (20) and (21) can be written as

$$\frac{1}{\alpha}\frac{d}{dt} q_r(t) = -2q_r(t) + \frac{\lambda}{d}\sum_\delta q_{r+\delta}(t) + b(t)\delta_{r,0}, \tag{23}$$

for all values of r.

Equation (23) can be examined from two points of view. On the one hand, it is a first-order differential equation in time. On the other hand, the summation over nearest neighbors resembles a discrete lattice Laplacian, and from this point of view it is a discrete second-order difference equation that can be solved by the use of a Green function. We then introduce the Laplace transform of $q_r(t)$,

$$\hat{q}_r(s) = \int_0^\infty dt e^{-st} q_r(t), \tag{24}$$

so that Eq. (23) may be written as

$$\frac{1}{\alpha}[-m_0^2 + s\hat{q}_r(s)] = -2\hat{q}_r(s) + \frac{\lambda}{d}\sum_\delta \hat{q}_{r+\delta}(s) + \hat{b}(s)\delta_{r,0}, \tag{25}$$

where $\hat{b}(s)$ is chosen such that $q_0(t)=1$, which is equivalent to taking $\hat{q}_0(s)=1/s$. For a random initial condition, which corresponds to a quench from infinite temperature, it is appropriate to take $q_r(0)=m_0^2(1-\delta_{r,0})+\delta_{r,0}$.

To solve Eq. (25), we introduce the lattice Green function associated with a d-dimensional hypercubic lattice,

$$\hat{G}_r(s) = \alpha \int \frac{e^{ikr}}{s + 2\alpha f(k)} \frac{d^d k}{(2\pi)^d}, \tag{26}$$

where the integration is over the first Brillouin zone, and

$$f(k) = 1 - \frac{\lambda}{d}\sum_{j=1}^{d} \cos k_j. \quad (27)$$

This Green function satisfies

$$\frac{1}{\alpha} s\hat{G}_r(s) = -2\hat{G}_r(s) + \frac{\lambda}{d}\sum_{\delta} \hat{G}_{r+\delta}(s) + \delta_{r,0}. \quad (28)$$

In terms of this lattice Green function, the solution of Eq. (15) can be written as

$$\hat{q}_r(s) = \frac{m_0^2}{s + 2\alpha(1-\lambda)} + \hat{b}(s)\hat{G}_r(s), \quad (29)$$

where $\hat{b}(s)$ should be chosen so that $\hat{q}_0(s) = 1/s$. This leads to

$$\hat{b}(s) = \frac{1}{\hat{G}_0(s)}\left[\frac{1}{s} - \frac{m_0^2}{s + 2\alpha(1-\lambda)}\right], \quad (30)$$

from which follows [3] the solution

$$\hat{q}_r(s) = \frac{m_0^2}{s + 2\alpha(1-\lambda)}\left[1 - \frac{\hat{G}_r(s)}{\hat{G}_0(s)}\right] + \frac{\hat{G}_r(s)}{s\hat{G}_0(s)}. \quad (31)$$

For a completely random initial condition, $m_0 = 0$, we have $m(t) = 0$,

$$\hat{b}(s) = \frac{1}{s\hat{G}_0(s)}, \quad (32)$$

and

$$\hat{q}_r(s) = \frac{\hat{G}_r(s)}{s\hat{G}_0(s)}. \quad (33)$$

III. TWO-TIME RESPONSE AND AUTOCORRELATION FUNCTIONS

The calculation of the two-time response function,

$$R(t,t') = \frac{1}{N}\sum_{r\in\Lambda} \left.\frac{\delta m_r(t)}{\delta h_r(t')}\right|_{h\downarrow 0}, \quad (34)$$

requires the application of a small perturbation, which is introduced according to the prescription (4), and from which one measures the response of the system. As pointed out in Sec. I, we assume a perturbed spin-flip rate, given by

$$w_r(\sigma) = \frac{\alpha}{2}\left[1 - \frac{\lambda}{2d}\sigma_r \sum_{\delta} \sigma_{r+\delta}\right][1 - h_r(t)\sigma_r], \quad (35)$$

where $h_r(t)$ is a time-dependent disturbance coupled to the dynamic variable $\sigma_r(t)$. We then write the equation of motion for the site magnetization,

$$\frac{1}{\alpha}\frac{d}{dt}m_r(t) = -m_r(t) + \frac{\lambda}{2d}\sum_{\delta} m_{r+\delta}(t) + h_r(t)\left[1 - \frac{\lambda}{2d}\sum_{\delta} q_{\delta}(t)\right], \quad (36)$$

which can also be written as

$$\frac{1}{\alpha}\frac{d}{dt}m_r(t) = -m_r(t) + \frac{\lambda}{2d}\sum_{\delta} m_{r+\delta}(t) + \frac{1}{2}h_r(t)b(t), \quad (37)$$

where $b(t)$ is given by Eq. (22).

Using the same procedures adopted to solve Eq. (10), and taking into account that the last term in Eq. (37) is a known function of time, we have

$$m_r(t) = \sum_{r'} \Gamma_{r-r'}(t)m_{r'}(0) + \frac{\alpha}{2}\sum_{r'} \int_0^t \Gamma_{r-r'}(t-t')h_{r'}(t')b(t')dt'. \quad (38)$$

From this expression, we calculate

$$\frac{\delta m_r(t)}{\delta h_{r'}(t')} = \frac{\alpha}{2}\Gamma_{r-r'}(t-t')b(t'), \quad (39)$$

which leads to the response function,

$$R(t,t') = \frac{\alpha}{2}\Gamma_0(t-t')b(t'). \quad (40)$$

The correlation $\langle\sigma_r(t)\sigma_r(t')\rangle$ of a spin at a given site r, at time t', with the same spin at a later time $t(t \geqslant t')$ is formally written as

$$\langle\sigma_r(t)\sigma_r(t')\rangle = \sum_{\sigma}\sum_{\sigma'} \sigma_r(t)P(\sigma,t|\sigma',t')\sigma_r'(t')P(\sigma',t'), \quad (41)$$

where $P(\sigma,t|\sigma',t')$ is the conditional probability of finding the configuration σ at time t given the configuration σ' at an earlier time t'. Noting that the site magnetization, $m_r(t) = \langle\sigma_r(t)\rangle$, with the initial condition $m_r(t') = \sigma_r'(t')$, may be written as

$$\sum_{\sigma} \sigma_r P(\sigma,t|\sigma',t') = m_r(t), \quad (42)$$

and using Eq. (19), we have

$$m_r(t) = \sum_{r'} \Gamma_{r-r'}(t-t')\sigma_{r'}(t'), \quad (43)$$

which can be inserted into Eq. (41) to give

$$\langle\sigma_r(t)\sigma_r(t')\rangle = \sum_{\sigma'}\sum_{r'} \Gamma_{r-r'}(t-t')\sigma_{r'}'(t')\sigma_r' P(\sigma',t'), \quad (44)$$

which finally leads to

$$C(t,t') = \sum_{r} \Gamma_r(t-t')q_r(t'). \quad (45)$$

A. Stationary regime

In the stationary regime the waiting time t' and the observation time t grow without limits, but the difference $t-t'$ is fixed. To be more precise, $t' \to \infty$, with $t \geqslant t'$ and $\tau = t - t'$ fixed.

From the Laplace transform final value theorem, we have

$$q_r(\infty) = \lim_{t\to\infty} q_r(t) = \lim_{s\to 0} s\hat{q}_r(s) = \frac{\hat{G}_r(0)}{\hat{G}_0(0)}, \tag{46}$$

so that

$$C(t,t') = C(\tau) = \sum_r \Gamma_r(\tau)\frac{\hat{G}_r(0)}{\hat{G}_0(0)}, \tag{47}$$

which can be written as

$$C(\tau) = \frac{1}{\hat{G}_0(0)}\int \frac{1}{2f(k)} e^{-\alpha f(k)\tau}\frac{d^dk}{(2\pi)^d}. \tag{48}$$

On the other hand, taking into account that

$$b(\infty) = \lim_{t\to\infty} b(t) = \lim_{s\to 0} s\hat{b}(s) = \frac{1}{\hat{G}_0(0)}, \tag{49}$$

the response function (40) can be written as

$$R(t,t') = R(\tau) = \frac{\alpha}{2}\Gamma_0(\tau)\frac{1}{\hat{G}_0(0)}. \tag{50}$$

Using the definition of $\Gamma_r(t)$, given by Eq. (18), we have

$$R(\tau) = \frac{\alpha}{2\hat{G}_0(0)}\int e^{-\alpha f(k)\tau}\frac{d^dk}{(2\pi)^d}. \tag{51}$$

Both quantities, $C(\tau)$ and $R(\tau)$, are time-translationally invariant (functions of τ only), as should be anticipated in a stationary regime. Moreover, the fluctuation-dissipation theorem is trivially satisfied, with $R(\tau) = -dC(\tau)/d\tau$. For large time differences, and $\lambda \neq 1$, it is easy to see that both the autocorrelation and the response functions decay exponentially, according to $\exp[-\alpha(1-\lambda)\tau]$, with the equilibration time $\tau_{\text{eq}} = 1/[\alpha(1-\lambda)]$.

B. Global fluctuation-dissipation relation

The fluctuation-dissipation theorem can also be written in terms of global variables, as the magnetization and the corresponding answer to a static perturbation. Let M denote the magnetization,

$$M(t) = \frac{1}{N}\sum_r \langle \sigma_r(t)\rangle, \tag{52}$$

and let us consider a static homogeneous disturbance h, defined by Eq. (3). Then

$$\frac{dM(t)}{dh} = \chi(t), \tag{53}$$

where $\chi(t)$ is a variance,

$$\chi(t) = \frac{1}{N}\sum_r \sum_{r'} [\langle \sigma_r(t)\sigma_{r'}(t)\rangle - \langle \sigma_r(t)\rangle\langle \sigma_{r'}(t)\rangle]. \tag{54}$$

Due to the translational invariance of the lattice, $M(t) = \langle \sigma_0(t)\rangle$. This quantity is the solution of Eq. (37), which is now given by

$$\frac{1}{\alpha}\frac{d}{dt}M(t) = -(1-\lambda)M(t) + \frac{1}{2}hb(t), \tag{55}$$

with the stationary solution

$$M = h\frac{b(\infty)}{2(1-\lambda)} = h\frac{1}{2(1-\lambda)\hat{G}_0(0)}. \tag{56}$$

For $\lambda < 1$, the magnetization vanishes as $h \to 0$, so that the variance is given by

$$\chi(t) = \sum_r q_r(t). \tag{57}$$

From Eq. (23) it follows that

$$\frac{1}{\alpha}\frac{d}{dt}\chi(t) = -2(1-\lambda)\chi(t) + b(t), \tag{58}$$

with the stationary solution

$$\chi = \frac{b(\infty)}{2(1-\lambda)} = \frac{1}{2(1-\lambda)\hat{G}_0(0)}. \tag{59}$$

From Eqs. (56) and (59), it is seen that relation (53) is clearly satisfied in the stationary regime.

C. Aging regime

The temporal behavior of the autocorrelation and the response functions, as calculated in this section, already suggests the existence of an aging regime. Equations (40) and (45), for $R(t,t')$ and $C(t,t')$, are valid for all values of t' and t, with $t - t' = \tau \geq 0$. The dependence of these functions on both t and t', and not on τ only, leads to the existence of aging. In this regime, the role of the spatial correlations is still crucial, since they are responsible for the realization of the aging scenario. In the stationary regime, it should be noted that q_r becomes a time-independent quantity in the $t' \to \infty$ limit only, and this is the reason for the dependence of the two-time functions on τ only.

In the transient regime, we do not expect the validity of the usual form of the fluctuation-dissipation relation given by Eq. (5). It has been convenient [10,11,2,12,13] to characterize the distance to the stationary regime by the fluctuation-dissipation ratio,

$$X(t,t') = \frac{R(t,t')}{\partial C(t,t')/\partial t'}. \tag{60}$$

A particular interesting quantity is the limit

$$X_\infty = \lim_{t'\to\infty} X(\infty,t'), \tag{61}$$

where

$$X(\infty,t') = \lim_{t\to\infty} X(t,t'). \tag{62}$$

From the results of this section, it is easy to write

$$X(\infty,t') = \frac{b(t')}{2b(t') - 2(1-\lambda)\chi(t')}, \tag{63}$$

where $\chi(t')$ is the sum given by Eq. (57). At the critical point, $\lambda=1$, it follows that $X(\infty,t')=1/2$ for any time t', so that $X_\infty=1/2$. In the disordered phase, $\lambda \neq 1$, Eqs. (58) and (59) may be used to conclude that $X_\infty=1$.

IV. CONCLUSIONS

We have reported a number of exact calculations for the dynamical behavior of a d-dimensional linearized version of the stochastic Glauber model. In one dimension, both the linear and the original model are essentially equivalent. For $d \geqslant 2$, however, the rates of transition of the linear Glauber model do not obey the conditions of detailed balance. This linear model can be regarded as a voter model with noise; it displays just one stable phase as long as noise is finite, and becomes critical in the absence of noise ($\lambda=1$). Since the dynamical properties are usually conceived for systems that do obey detailed balance, we decided that it was appropriate to carry out some explicit calculations for a microscopically irreversible model.

We have obtained expressions for the two-time autocorrelation, $C(t,t')$, and response functions, $R(t,t')$, which depend on both observation time t and waiting time $t' \leqslant t$. In the stationary, infinite time limit, $t' \to \infty$, in the presence of noise ($0<\lambda<1$), the spatial correlations are independent of time, and the two-time functions become translationally invariant (depending on the difference $\tau=t-t'$ only). The usual form of the fluctuation-dissipation theorem, $X(t,t') = R(t,t')/\partial C(t,t')/\partial t'=1$, is trivially observed in this regime.

In the scaling regime ($t\to\infty$), for $0\leqslant\lambda<1$, we obtain a nontrivial fluctuation-dissipation ratio, $X(\infty,t')\neq 1$. At the critical point, $\lambda=1$, we have $X(\infty,t')=1/2$, which further indicates that the dynamical behavior of the d-dimensional microscopically irreversible linear Glauber model is very similar to the behavior of its one-dimensional reversible counterpart.

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